

COHOMOLOGY AND QUANTUM GROUPS

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Preamble.

Section 1 is an abbreviated version of a preprint with the same title by C. Frønsdal and A. Galindo [17]. A new and unpublished result by the same authors can be reported here. The representations of the braid group that are obtained from R-matrices associated with multiparameter quantum $gl(N)$ are factored through the Hecke algebra. In other words, the spectrum of the generator consists of only two points; we refer to this as the Hecke condition. It turns out that the deformations that preserve the braid relation (Yang-Baxter) automatically preserve the Hecke condition. The feature of the R-matrix that naturally associates it to $gl(n)$ is thus preserved by deformations. Section 2 is an alternative determination of the classical r-matrices for simple Lie algebras first found by Belavin and Drinfeld. The methodology consists of calculating the deformations of the simplest class of coboundary Lie bialgebras (those that respect the Cartan subalgebra and are related to the twisted quantum groups). They are mostly rigid to essential deformations, but on special surfaces in the space of parameters one finds essential deformations that turn out to reproduce the entire panorama of Belavin-Drinfeld r-matrices. Section 3 reviews the tools used, cohomology on Lie algebras and the double complex on Lie bialgebras. There follows an application to the Manin triplet; we prove that, as a Lie algebra, $\mathfrak{g} \bowtie \mathfrak{g}^*$ is isomorphic to $\mathfrak{g} \oplus \mathfrak{g}$ in all cases. Section 4 deals with quantization and returns to the problem of deformations of quantum groups in the general case. A lecture on the Universal T-matrix, in which a solution was offered to the problem of exponentiation on quantum groups, is not covered by these notes. The material may be found in ref.10.

1. Deformations of Quantum $gl(n)$.

A. MULTIPARAMETER QUANTUM $gl(N)$.

Belavin and Drinfeld [1] classified the r -matrices (structures of coboundary Lie bialgebra) associated to simple Lie algebras, both finite and affine. A program of “quantization” of Lie algebras proposed by Drinfeld [2] would promote the classical structures to bialgebra deformations of enveloping algebras. Standard forms of such “quantum deformations” of the simple affine Lie algebras were obtained by Jimbo [3]. Here we are interested only in the finite case; that is, constant r - and R -matrices. So far there is no general classification of quantized Lie algebras.

Deformation theory applied at the classical point [4] is difficult, since the obstructions appear only in the second order. But a large family of exact quantum deformations of $gl(N)$ is known [5], with $1 + N(N - 1)/2$ parameters. These algebras are rigid (with respect to essential deformations) at generic points in parameter space, even to first order deformations, but essential deformations exist on algebraic surfaces of lower dimension, for $N > 2$. The determination of all the first order deformations, presented here, goes far towards a complete classification of all formal and/or exact deformations. All first order deformations are combinations of “elementary” deformations, and all the elementary deformations are exact. [A large class of exact deformations were described in ref.9; I learned at this School that a one-parameter family belonging to the same class had already been discovered by Cremmer and Gervais, ref.18.]

We consider the free associative algebra \mathcal{F}_x generated by (x^i) , $i = \dots, N$, and the ideal \mathcal{F}_{x0} generated by

$$x^i x^j - q^{ij} x^j x^i, \quad i, j = 1, \dots, N, \quad (1.1)$$

in which the q 's are taken from a field K with characteristic 0, with

$$q^{ij} q^{ji} = 1, \quad q^{ii} = 1, \quad i, j = 1, \dots, N. \quad (1.2)$$

We call **quantum plane** the associative algebra $\mathcal{F}_x/\mathcal{F}_{x0}$; that is, the associative algebra generated by the x 's with relations

$$x^i x^j - q^{ij} x^j x^i = 0, \quad i, j = 1, \dots, N. \quad (1.3)$$

Similarly, the quantum anti-plane $\mathcal{F}_\theta/\mathcal{F}_{\theta 0}$ is an associative algebra generated by N elements (θ^i) , $i = 1, \dots, N$, with relations

$$\theta^i \theta^j + r^{ij} \theta^j \theta^i = 0, \quad i, j = 1, \dots, N, \quad (1.4)$$

in which the r 's are parameters from K satisfying the same relations as the q 's, Eq. (1.2). Let V denote the linear vector space over K spanned by the x 's (or by the θ 's).

Definition. A **generalized symmetry** is an element P of $\text{End}(V \otimes V)$ that satisfies the Hecke condition

$$(P - 1)(P + a) = 0, \quad (1.5)$$

for some $a \in K, a \neq -1, 0$. We shall say that the tensor $xx = (x^i x^j)$ is P -symmetric, and that the tensor $\theta\theta = (\theta^i \theta^j)$ is P -antisymmetric, if

$$xx(P - 1) = 0, \quad \theta\theta(P + a) = 0. \quad (1.3'-4')$$

Let P_{12} be the operator on $V \otimes V \otimes V$ that acts as P on the two first factors; the braid relation is

$$P_{12}P_{23}P_{12} = P_{23}P_{12}P_{23}. \quad (1.6)$$

Theorem 1. Given relations (1.3) and (1.4), with parameters q and r subject to the conditions (1.2), the following two statements are equivalent:

$$(a) \quad r^{ij} = aq^{ij}, \quad i < j, \quad i, j = 1, \dots, N; \quad (1.7)$$

(b) There exists P in $\text{End}(V \otimes V)$ satisfying (1.5-6), such that the relations (1.3-4) coincide respectively with Eqs. (1.3'-4'); it is unique up to a permutation of the basis.

Let P be a generalized symmetry of dimension N . Consider the algebra \mathcal{F}_x freely generated by (x^i) , $i = 1, \dots, N$, with the ideal \mathcal{F}_{x0} generated by

$$(xx(P - 1))^{ij}, \quad i, j = 1, \dots, N; \quad (1.8)$$

and the algebra \mathcal{F}_θ generated by (θ^i) $i = 1, \dots, N$, with the ideal $\mathcal{F}_{\theta 0}$ generated by

$$\theta\theta(P + a). \quad (1.9)$$

Let \mathcal{F} be the algebra generated by (x^i) and (θ^i) , $i = 1, \dots, N$ with relations

$$ax\theta = \theta xP. \quad (1.10)$$

This algebra contains \mathcal{F}_x and \mathcal{F}_θ as subalgebras and the ideals \mathcal{F}_{x0} and $\mathcal{F}_{\theta0}$ are thus canonically identified with subsets of \mathcal{F} .

Theorem 2. Suppose that the generalized symmetry P satisfies the braid relation. Let X denote the linear span of the x 's and Θ the linear span of the θ 's, then the statements

$$\mathcal{F}_{x0}\Theta = \Theta\mathcal{F}_{x0} \quad (1.11)$$

$$\mathcal{F}_{\theta0}X = X\mathcal{F}_{\theta0} \quad (1.12)$$

hold in \mathcal{F} . Conversely, if both (1.11) and (1.12) hold, then P satisfies the braid relation.

Proof. Eqs. (1.11), (1.12) are equivalent, respectively, to

$$(\text{braid})_{123}(P_{12} - 1) = 0, \quad (\text{braid})_{123}(P_{12} + a) = 0, \quad (1.13)$$

with

$$(\text{braid})_{123} := P_{12}P_{23}P_{12} - P_{23}P_{12}P_{23}. \quad (1.14)$$

Definition. Let $\langle q \rangle$ stand for a set of parameters (q^{ij}) , $i, j = 1, \dots, N$, satisfying $q^{ij}q^{ji} = 1$ and $q^{ii} = 1$; and a an additional parameter, all in the field K . Let $r^{ij} = aq^{ij}$ for $i < j$, $r^{ii} = 1$ and $r^{ji} = q^{ji}/a$ for $i < j$. The **standard quantum algebra** $A(\langle q \rangle, a)$ is generated by the x 's and the θ 's, with relations (1.3-4) and (1.10). More generally, for any generalized symmetry P , the algebra $A(P)$ generated by the x 's and the θ 's, with relations (1.3'-4') and (1.10), will be called a **quantum P-algebra** if the conditions (1.11) and (1.12) hold.

The purpose of this paper is to study deformations of $A(\langle q \rangle, a)$ in the category of quantum P-algebras. The quantum pseudogroup (in the sense of Woronowicz [6]) associated to P is the unital algebra generated by the matrix elements of an N -by- N matrix T , with relations

$$[P, T \otimes T] = 0. \quad (1.15)$$

It is the algebra of linear automorphisms of $A(P)$; that is, the set of mappings

$$(x, \theta, T) \rightarrow x \otimes T, \quad \theta \otimes T \quad (1.16)$$

that preserve the relations (1.3'-4', 1.10) of $A(P)$. It is related, via duality, to a quantum group in the sense of Drinfeld [2]. Twisted, quantum $gl(N)$ [5] corresponds to $\text{Aut}A(\langle q \rangle, a)$. The deformations of this quantum group are in 1-to-1 correspondence with the deformations of the standard quantum algebra $A(\langle q \rangle, a)$.

The next section defines the deformations that will be calculated in this paper. It turns out that $A(\langle q \rangle, a)$ is rigid for parameters in general position. Interesting nontrivial deformations (even exact ones) do exist on certain algebraic surfaces in parameter space, for $N > 2$. The existence of an unexpected special case that requires $a^3 = 1$ deserves some attention.

B. DEFORMATIONS.

Henceforth, P will denote the generalized symmetry associated with $A(\langle q \rangle, a)$. A **formal deformation** of $A(\langle q \rangle, a)$ is here a quantum $P(\epsilon)$ -algebra with $P(\epsilon)$ a formal power series in an indeterminate ϵ

$$P(\epsilon) = P + \epsilon P_1 + \epsilon^2 P_2 + \cdots, \quad (1.17)$$

that satisfies the Hecke condition with the parameter a independent of ϵ , and such that (1.11-12) hold. In this case we shall say that $P(\epsilon)$ is a formal deformation of P . A deformation is **exact** if the series $P(\epsilon)$ has a nonvanishing radius of convergence. If $P(\epsilon)$ is a formal deformation of P , then

$$P(\epsilon, 1) = P + \epsilon P_1 \quad (1.18)$$

is a first order deformation. More generally, a **first order deformation** is defined as a formal deformation except that one sets $\epsilon^2 = 0$. A first order deformation is not necessarily the first two terms of a formal deformation. For example, at the classical point, where a and all the q 's are equal to unity, the braid relation is moot in first order. (Recall that the condition that defines a formal deformation is in this case the classical Yang-Baxter relation, which is second order.) For this reason, the concept of a first order deformation is of no use at the classical point. In contrast with this, we shall find that, at general position in parameter space, $A(\langle q \rangle, a)$ is rigid with respect to first order deformations, which implies rigidity under formal deformations. The case $a = 1$, with the q 's in general position, is in this respect intermediate; it is best treated separately. Two types of deformations (and combinations of them) will be considered trivial. A linear transformation, with coefficients in $K[\epsilon]$,

$$x^i \rightarrow x^i + \epsilon x^j A_j^i + \dots, \quad \theta^i \rightarrow \theta^i + \epsilon \theta^j A_j^i + \dots, \quad (1.19)$$

induces a trivial, formal deformation of P . A variation of the q 's

$$q^{ij} \rightarrow q^{ij} + \epsilon \delta q^{ij} + \dots \quad (1.20)$$

will also be considered trivial. A deformation that is not trivial is called essential. We shall classify the equivalence classes, with respect to the transformations (1.19-20), of first order deformations.

Theorem 3. If $a \neq 1$, then each equivalence class of first order deformations contains a unique representative with the property that $(P_1)_{kl}^{ij} = 0$ for every index set i, j, k, l that contains no more than two different numbers.

The first order deformation of P induced by (1.19) is

$$P_1 = PZ - ZP, \quad Z := A \otimes 1 + 1 \otimes A, \quad (1.21)$$

or more explicitly

$$(P_1)_{kl}^{ij} = a(\hat{q}^{lk} - \hat{q}^{ij})Z_{kl}^{ji} + (1 - a)[(k < l) - (i < j)]Z_{kl}^{ij}, \quad (1.22)$$

where

$$\hat{q}^{ij} := \begin{cases} q^{ij} & \text{if } i < j, \\ q^{ij}/a & \text{if } i \geq j, \end{cases} \quad (i < j) := \begin{cases} 1 & \text{if } i < j, \\ 0 & \text{otherwise.} \end{cases} \quad (1.23)$$

Preservation of the Hecke condition (1.5) under first order deformations is equivalent to requiring that

$$\begin{aligned} q^{ij}(P_1)_{lk}^{ji} + \hat{q}^{kl}(P_1)_{kl}^{ij} &= 0 \\ (1-a)q^{ij}(P_1)_{lk}^{ji} &= (a-1)\hat{q}^{kl}(P_1)_{kl}^{ij} = (P_1)_{lk}^{ij} + aq^{ij}\hat{q}^{kl}(P_1)_{kl}^{ji}, \quad i \leq j, \quad k \leq l. \end{aligned} \quad (1.24)$$

The main difficulty is to extract the conditions on P_1 imposed by the braid relation. We found that the strategy made available by Theorem 2 simplifies this task. Both conditions, (1.11) and (1.12), must be invoked. We leave out the details.

Definition. An elementary first order deformation is one in which some $(P_1)_{kl}^{ij}$ is non-zero for just one unordered pair i, j and just one unordered pair k, l .

Theorem 4. Suppose $a \neq \pm 1, 0$. There are two series of elementary, first order, essential deformations. The “principal series” is described first. Let i, j be any index pair with, either (case 1) $k+1 = i \leq j = l-1$, or else (case 2) $i+1 = k \leq l = j-1$. Let $P_1 = 0$, except that

$$(P_1)_{lk}^{ij} = -a\hat{q}^{ij}q^{kl}(P_1)_{kl}^{ji} \neq 0. \quad (1.25)$$

This defines an elementary deformation if and only if the parameters satisfy the conditions

$$q^{im}q^{jm}q^{mk}q^{ml} = \begin{cases} a^x, & \text{case 1,} \\ a^{-x}, & \text{case 2;} \end{cases} \quad x = \delta_m^i - \delta_m^j, \quad m = 1, 2 \dots N. \quad (1.26)$$

The “exceptional series” of elementary, first order deformations exists only if $a^3 = 1$. Let i, j, k be neighbors in the natural numbers, with $i+1 = j$. Let $P_1 = 0$, except that either $(P_1)_{kk}^{ij} = -aq^{ij}(P_1)_{kk}^{ji} \neq 0$, or else $(P_1)_{ij}^{kk} = -q^{ij}(P_1)_{ji}^{kk} \neq 0$, but not both. This defines an elementary deformation if and only if the parameters satisfy

$$\begin{aligned} (P_1)_{kk}^{ij} \neq 0 : \quad & (q^{km})^2 q^{mj} q^{mi} = a^x, \quad x = \delta_{mi} - \delta_{mj}; \\ (P_1)_{ij}^{kk} \neq 0 : \quad & (q^{km})^2 q^{mj} q^{mi} = a^x, \quad x = \pm(\delta_{mk} - \delta_{mi}). \end{aligned} \quad (1.27)$$

The two signs in the last line apply when $k = i-1, k = j+1$, respectively. There are no other first order, elementary deformations. The elementary deformations are formal and exact.

C. THE CLASSICAL LIMIT.

All the results obtained here, for twisted quantum $gl(N)$, have direct application to twisted, quantum $sl(N)$. The connection between these two was explained by Schirrmacher in [5] and is discussed also in [10]. In this section we shall take the classical limit and confront our results for $gl(N)$ with the classification, by Belavin and Drinfeld [1], of the classical r-matrices for $sl(N)$. Strictly, this is possible only under additional conditions on the parameters, namely

$$\prod_i q^{ij} a^j = a^{(N+1)/2}; \quad (1.28)$$

one may therefore assume that these relations hold, although they do not interfere directly with the following calculations.

The deformed quantum P-algebras of the principal series are semiclassical. The classical r-matrix is defined by expanding the parameters,

$$a = 1 + h, \quad q^{ij} = 1 + hp^{ij}, \quad i < j, \quad (1.29)$$

and the R-matrix,

$$R_{lk}^{ij} := P_{kl}^{ij} + \epsilon(P_1)_{kl}^{ij}, \quad (1.30)$$

in powers of h ,

$$R = 1 - hr_\epsilon + O(h^2), \quad r_\epsilon = r + \epsilon\delta r. \quad (1.31)$$

Here r is the r-matrix for twisted (= multiparameter) $gl(N)$,

$$r = \sum_{i < j} M_j^i \otimes M_i^j + r_0, \quad (1.32)$$

with

$$r_0 := \sum_{i < j} (p^{ij} M_j^j \otimes M_i^i - (1 + p^{ij}) M_i^i \otimes M_j^j), \quad (1.35)$$

and the perturbation is

$$h\delta r_{kl}^{ij} = (P_1)_{lk}^{ij}, \quad \text{or} \quad h\delta r = \sum_{i,j,k,l} (P_1)_{lk}^{ij} M_k^i \otimes M_l^j. \quad (1.34)$$

Here M_k^j is the matrix with the unit in row- k , column- j , all the rest zero.

We examine the classical limit of an elementary, first-order deformation. Fix the notation as in Theorem 4, the expression for δr is, up to a constant,

$$\delta r = M_k^i \otimes M_l^j - M_l^j \otimes M_k^i. \quad (1.35)$$

The diagonal matrices (M_i^i) , $i = 1, \dots, N$, will be taken as a basis for a “Cartan subalgebra of $gl(N)$ ”. The upper triangular matrices form the subspace of positive roots and the matrices M_i^j with $i - j = \pm 1$ are the simple roots. [We have abused the notation by extending the notion of roots from $sl(n)$ to $gl(n)$ and by introducing both positive and negative “simple” roots.] The conditions on the indices that are spelled out in Theorem 4 insure that all the roots appearing in (1.35) are simple, and that a positive root is paired with a negative root and *vice versa*. Let us find out what is the meaning of the restriction (1.26) on the parameters.

Let $\Gamma_1(\Gamma_2)$ be the root space spanned by $M_k^i(M_j^l)$ and $\tau : \Gamma_1 \rightarrow \Gamma_2$ the mapping defined by $\tau(M_k^i) = M_j^l$. Consider the equation

$$(\tau\alpha \otimes 1 + 1 \otimes \alpha)r_0 = 0, \quad (1.36)$$

where $\alpha = M_k^i$ and $(\alpha \otimes 1)H \otimes H' = \alpha(H)H'$ and $(1 \otimes \alpha)H \otimes H' = \alpha(H')H$, H and H' in the diagonal subalgebra of $gl(N)$. We find that this equation is equivalent to

$$p^{lm} + p^{km} + p^{mi} + p^{mj} = \delta_m^j - \delta_m^i, \quad (1.37)$$

This equation is precisely the first order analog of Eq. (1.26), while (1.36) is a condition (Eq. (6.7)) of Belavin and Drinfeld [1]. We have thus established that the conditions (1.26) on the parameters lift the invariance condition of ref. [1] to the quantum algebra.

2. Deformation of twisted, simple Lie bialgebras.

A. THE PROBLEM.

The (constant) classical Yang-Baxter matrices for simple Lie algebras were already classified by Belavin and Drinfeld [1]. Here we shall approach the same problem by the methods of deformation theory; we shall see that the calculations are straightforward and that this method throws some light on the classification. The results show a clear relationship to the deformations of twisted quantum, $gl(n)$ and suggest that the deformations of all simple quantum groups can be obtained with less effort using cohomological methods.

This lecture merely shows the calculations; therefore everything is referred to a basis. The theoretical background is explained in Section 3.

Let \mathfrak{g} be a simple Lie bialgebra over \mathbb{C} , (L_i) a basis, structure tensors ϵ and f :

$$[L_i, L_j] = \epsilon_{ij} = \epsilon_{ij}^k L_k, \quad \Delta(L_i) = f_i = f_i^{jk} L_j \otimes L_k.$$

Here ϵ is a evidently a two-form valued in \mathfrak{g} , satisfying the Jacobi identity

$$(d\epsilon)_{ijk} = \sum_{(ijk)} \epsilon_{im}^n \epsilon_{jk}^m = 0,$$

and f is a oneform valued in $\mathfrak{g} \otimes \mathfrak{g}$. The cohomology operator of \mathfrak{g} is denoted d . The “compatibility condition” between the two structures reads

$$df = 0, \quad (df)_{ij} = L_i f_j - L_j f_i - \epsilon_{ij}^k f_k. \quad (2.1)$$

In the case of a coboundary Lie algebra f is exact,

$$f = dr, \quad f_i = [L_i, r], \quad r = r^{ij} L_i \otimes L_j.$$

From now on we set $f = dr$. Let \mathfrak{g}^* denote the vector space dual of \mathfrak{g} , dual basis Γ^i ; it becomes a Lie bialgebra with structure tensors f and ϵ ,

$$\{\Gamma^i, \Gamma^j\} = f^{ij} = f_k^{ij} \Gamma^k, \quad \Delta(\Gamma^i) = \epsilon^i = \epsilon_{jk}^i \Gamma^j \otimes \Gamma^k.$$

The cohomology operator on the Lie algebra \mathfrak{g}^* is denoted ∂ . (Section 3.) The Jacobi identity for the Lie bracket $\{, \}$ of \mathfrak{g}^* is

$$\partial f = 0, \quad (\partial f)_m = f_m^{in} f_n^{jk} L_i \wedge L_j \wedge L_k.$$

Since $f = dr$, and since d and ∂ anticommute, it may be expressed in terms of the classical Yang-Baxter or Schouten bracket,

$$\partial f = \partial dr = -d\partial r = 0, \quad \partial r = f_n^{ij} r^{kn} L_i \wedge L_j \wedge L_k, \quad (2.2)$$

or

$$d(YB) = 0, \quad YB = \partial r = [r_{12}, r_{13} + r_{23}] + [r_{13}, r_{23}].$$

At the cost of including an invariant, symmetric piece in r we can take r to be ∂ -closed, $\partial r = 0$. Thus, following Belavin and Drinfeld, we pose

$$r + r^t = K, \quad \partial r = 0, \quad (2.3)$$

where K is the Killing form of \mathfrak{g} .

We can deform in the category of Lie bialgebras or in the category of boundary Lie bialgebras. In the first case we deform f , in the second case r . Since the symbol ϵ is used for the structure tensor we use \hbar for the deformation parameter.

To deform in the category of Lie bialgebras; one would set

$$f(\epsilon) = dr + \hbar f_1 + \dots,$$

and impose associativity and closure,

$$\partial f(\hbar) = 0, \quad df(\hbar) = 0.$$

To first order in \hbar (with $a, b, c \in \mathfrak{g}^*$)

$$\partial f_1(a, b, c) = \sum_{cyclic} (f_1(a, f(b, c)) + f(a, f_1(b, c))) = 0, \quad df_1 = 0.$$

The relevant cohomologies are $H^1(\mathfrak{g}, \mathfrak{g} \wedge \mathfrak{g})$, which is 0, and $H^2(\mathfrak{g}^*, \mathcal{C})$, which is not.

Deforming in the category of coboundary Lie algebras; we set

$$f(\hbar) = dr(\hbar), \quad r(\hbar) = r + \hbar r_1 + \dots, \quad (2.4)$$

and impose associativity only. To first order in \hbar ,

$$r_1 + r_1^t = 0, \quad \partial r_1 = 0,$$

or more explicitly

$$\partial r_1 = \sum f_m^{ij} r_1^{km} L_i \wedge L_j \wedge L_k = 0. \quad (2.5)$$

The cohomology is now $H^2(\mathfrak{g}^*)$ with coefficients in the field, since $r_1(a, b) \in K$. The problem is to calculate $H^2(\mathfrak{g}^*, \mathcal{C})$; that is, the equivalence classes of essential deformations, in the category of coboundary bialgebras, of the simplest Lie bialgebras (that correspond to the twisted quantum groups).

B. CALCULATION OF $H^2(\mathfrak{g}^*, \mathcal{C})$.

Notation. Choose a Cartan subalgebra \mathfrak{h} , a set of positive roots, and a Weyl basis $(e_\alpha, e_{-\alpha})_{\alpha > 0}, (h_i)$. The structure of \mathfrak{g} is

$$[h_i, e_\alpha] = r_i(\alpha) e_\alpha, \quad [e_\alpha, e_\beta] = e_{\alpha+\beta}, \quad \alpha < \beta, \quad [e_\alpha, e_{-\alpha}] = r^i(\alpha) h_i. \quad (2.6)$$

To make sense of this we must introduce an ordering on the root lattice; we assume that $\alpha < \beta$ implies that $-\alpha < -\beta$.

For r we have

$$r = \sum r_0^{ij} h_i \otimes h_j + \sum_{\alpha > 0} e_\alpha \otimes e_{-\alpha}, \quad (2.7)$$

and thus

$$r + r^t = \sum (r_0^{ij} + r_0^{ji}) h_i \otimes h_j + \sum_{\alpha} e_\alpha \otimes e_{-\alpha} = K^{ij} L_i \otimes L_j. \quad (2.8)$$

Here r^t is the transposed of r and the last expression is the invariant in $\mathfrak{g} \otimes \mathfrak{g}$. The coefficients define the Killing form K , and the restriction r_0 of r to \mathfrak{h} is

$$r_0 = \hat{r}_0 + (1/2)K_0, \quad (2.9)$$

with \hat{r}_0 antisymmetric and K_0 the restriction of K to \mathfrak{h} . Note that $r^j(\beta) = K_0^{ij} r_i(\beta)$.

We calculate the coproduct,

$$\Delta(L_i) = [L_i, r] = \sum f_i^{jk} L_j \wedge L_k, \quad (2.10)$$

and see that $\Delta(h_i) = 0$ while

$$\begin{aligned}
\Delta(e_\beta) &= [e_\beta, r] = \sum f_\beta^{ij} L_i \wedge L_j \\
&= \sum r_0^{ij} ([e_\beta, h_i] \otimes h_j + h_i \otimes [e_\beta, h_j]) + \sum_{\alpha > 0} ([e_\beta, e_\alpha] \otimes e_{-\alpha} + e_\alpha \otimes [e_\beta, e_{-\alpha}]) \\
&= -\hat{r}_0^{ij} r_i(\beta) e_\beta \wedge h_j + (1/2) \sum_{\alpha > 0} ([e_\beta, e_\alpha] \wedge e_{-\alpha} + e_\alpha \wedge [e_\beta, e_{-\alpha}]) \\
&= \sum \hat{r}_0^{ij} r_i(\beta) h_j \wedge e_\beta \\
&\quad + (1/2)(\beta < 0) r^j(\beta) h_j \wedge e_\beta + (1/2)(\beta > 0) r^j(\beta) e_\beta \wedge h_j + \dots,
\end{aligned}$$

The coefficients give us f and thus the Lie structure of \mathfrak{g}^* . The dual basis will be denoted $(x^\alpha, y^{-\alpha}, w^i)$. The structure of \mathfrak{g}^* is

$$\begin{aligned}
\{w^i, x^\alpha\} &= w^i(x^\alpha) x^\alpha, \quad \{w^i, y^{-\alpha}\} = w^i(y^{-\alpha}) y^{-\alpha}, \\
\{x^\alpha, y^{-\beta}\} &= 0, \quad \{x^\alpha, x^\beta\} = x^{\alpha+\beta}, \quad \{y^{-\alpha}, y^{-\beta}\} = y^{-\alpha-\beta}, \quad \alpha < \beta,
\end{aligned} \tag{2.11}$$

The “weights” $w^i(x^\alpha), \dots$ are in the field;

$$w^j(x_\beta) = \sum_i \left(\hat{r}_0^{ij} \mp (1/2) K_0^{ij} \right) r_i(\beta), \tag{2.12}$$

for β positive/negative.

Next, the differentials. The zero-forms are closed and never exact. Consequently, one-forms are never exact, $B^1 = 0, H^1 = Z^1 \subset C^1$. If A is a one-form,

$$A = A^i h_i + A^\alpha e_\alpha,$$

then

$$\partial A = f_k^{ij} A^k L_i \wedge L_j = 2 \sum w^i(\alpha) A^i h_i \wedge e_\alpha + \sum_{\alpha, \beta > 0} A^{\alpha+\beta} e_\alpha \wedge e_\beta. \tag{2.13}$$

So $\partial A = 0$ implies that

$$A = \sum_{simple} A^\alpha e_\alpha. \tag{2.14}$$

The sum is over simple roots α for which w does not vanish, which means all simple roots.

We turn to two-forms,

$$\begin{aligned}
B &= \sum B^{ij} L_i \wedge L_j, \\
dB &= \sum f_l^{ij} B^{kl} L_i \wedge L_j \wedge L_k \\
&= 2 \sum w^i(\alpha) B^{j,\alpha} h_i \wedge e_\alpha \wedge h_j + 2 \sum w^i(\alpha) B^{\beta,\alpha} h_i \wedge e_\alpha \wedge e_\beta \\
&\quad + \sum_{\alpha\beta>0} B^{i,\alpha+\beta} e_\alpha \wedge e_\beta \wedge h_i + \sum_{\alpha\beta>0} B^{\gamma,\alpha+\beta} e_\alpha \wedge e_\beta \wedge e_\gamma.
\end{aligned} \tag{2.15}$$

Here $\alpha\beta > 0$ means that the two roots are either both positive or both negative.

According to (2.13), a two-form is exact if

$$B^{ij} = 0, \quad B^{i\alpha} = w^i(\alpha) A^\alpha, \quad B^{\alpha\beta} = \begin{cases} A^{\alpha+\beta}, & \alpha\beta > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.16}$$

According to (2.15), a two-form is closed if

$$B^{i\alpha} = w^i(\alpha) A^\alpha, \quad B^{\alpha\beta} (w^i(\alpha) + w^i(\beta)) = \begin{cases} B^{i,\alpha+\beta}, & \alpha\beta > 0, \alpha, \beta \text{ simple} \\ 0, & \text{otherwise} \end{cases} \tag{2.17}$$

The result (2.12) for the weights shows that $w(\alpha) + w(\beta)$ cannot vanish unless $\alpha\beta < 0$. If α, β are both positive, then $w(\alpha) + w(-\beta) = 0$ is the same as

$$r^{ij} r_j(\alpha) + r_j(\beta) r^{ji} = 0. \tag{2.18}$$

Compare Eq.(1.36). From (2.16) and (2.17) we draw the following conclusion.

Proposition. The space of first order essential deformations of the bialgebra is

$$H^2 = Z^2/B^2 = \left\{ r_1 = \sum_{\sigma} r_1^{\alpha,\beta} e_\alpha \wedge e_\beta + \sum r_1^{ij} h_i \wedge h_j \right\},$$

where σ is the set

$$\sigma = \{(\alpha, \beta); \alpha, \beta \text{ simple}, w(\alpha) + w(\beta) = 0\};$$

and $(\alpha, \beta) \in \sigma$ implies that $\alpha\beta < 0$. The coefficients r_1^{ij} in the second term merely perturb the original parameters in r_0 and these deformations are considered trivial, to be ignored in the sequel.

C. THE HIGHER ORDERS.

To order \hbar^2 the closure condition is

$$d(YB(r_1) + \partial r_2) = 0,$$

or, since r_1, r_2 are antisymmetric,

$$YB(r_1) + \partial r_2 = 0.$$

The first term is, according to the Proposition (up to equivalence),

$$\begin{aligned} YB(r_1) = & (1/2) \sum r_1^{\alpha\beta} r_1^{\gamma\delta} ([e_\alpha, e_\gamma] \wedge e_{-\beta} \wedge e_{-\delta} + [e_{-\beta}, e_{-\delta}] \wedge e_\alpha \wedge e_\gamma \\ & - [e_\alpha, e_{-\delta}] \wedge e_{-\beta} \wedge e_\gamma - [e_{-\beta}, e_\gamma] \wedge e_\alpha \wedge e_{-\delta}). \end{aligned} \quad (2.19)$$

It is a general result of deformation theory that this quantity is ∂ -closed; now we have to find r_2 such that $\partial r_2 = -YB$, the obstructions to continuing the deformation are thus in $H^3(\mathfrak{g}^*, \mathcal{C})$. Note that the roots that appear in (2.19) are all simple; therefore

$$[e_\alpha, e_{-\delta}] = \delta_\alpha^\delta r^i(e_\alpha) h_i, \quad [e_{-\beta}, e_\gamma] = -\delta_\beta^\gamma r^i(e_\gamma) h_i.$$

One part of the YB bracket is thus

$$\begin{aligned} & - \sum r_1^{\alpha\beta} r_1^{\gamma\alpha} r^i(\alpha) h_i \wedge e_{-\beta} \wedge e_\gamma + \sum r_1^{\alpha\beta} r_1^{\beta\delta} r^i(\beta) h_i \wedge e_\alpha \wedge e_{-\delta} \\ & = 2 \sum r_1^{\alpha\beta} r_1^{\beta\delta} r^i(\gamma) h_i \wedge e_\alpha \wedge e_{-\delta}. \end{aligned} \quad (2.20)$$

To cancel this we need a term r_{21} in r_2 that satisfies

$$r_{21}^{\alpha, -\delta} (w^i(\alpha) + w^i(-\delta)) = 2 \sum_\beta r_1^{\alpha\beta} r_1^{\beta\delta} r^1(\gamma), \quad (2.21)$$

so that, if $w^i(\alpha) + w^i(-\delta) = 0$, then there would be an obstruction. But this is impossible; since $w(\alpha) + w(-\gamma) = 0$ it would imply that $w(-\gamma) - w(-\delta) = 0$, and thus $\gamma = \delta$; but $r^{\gamma\gamma} = 0$. [Remember that, in (2.21), the indices α, β, δ refer to simple roots.]

Turning to the remaining terms in (2.19), suppose that $\alpha + \gamma$ is a root, $\alpha + \gamma = \mu$. Then the first term in (2.19) becomes

$$\sum r_1^{\alpha\beta} r_1^{\gamma\delta} e_\mu \wedge e_{-\beta} \wedge e_{-\delta}.$$

To cancel it we need, as (2.15) shows, that two of the roots in this expression add up to a root. This can only be $-\beta - \delta = -\nu$. So here is an obstruction; if $\alpha + \gamma$ is a root and $\beta + \delta$ is not, and $r_1^{\alpha,\beta} r_1^{\gamma,\delta} \neq 0$, then there is no remedy. We must avoid this obstruction by imposing an additional condition on r_1 : in the formula

$$r_1 = \sum_{\sigma} r_1^{\alpha,\beta} e_{\alpha} \wedge e_{\beta},$$

if $r_1^{\alpha,\beta}$ and $r_1^{\gamma,\delta}$ are both not zero, and $\alpha + \gamma$ is a root, then $\beta + \delta$ must also be a root. The last two terms in (2.19) are thus

$$\sum r_1^{\alpha,\beta} r_1^{\beta,\delta} (e_{\alpha+\beta} \wedge e_{-\beta} \wedge e_{-\delta} + e_{-\beta-\delta} \wedge e_{\alpha} \wedge e_{\gamma}) = d(\sum r_{22}^{\alpha,\beta} e_{\alpha} \wedge e_{\beta})$$

with

$$r_{22}^{\alpha+\gamma, -\beta-\delta} = -r_1^{\alpha,\beta} r_1^{\gamma,\delta}. \quad (2.22)$$

The calculation can be carried on to higher orders, but the panorama of Belavin and Drinfeld [1] is already emerging and it may be sufficient to give the final answer.

Proposition. The exact deformations are classified up to equivalence by

$$r(\hbar) = r + \sum_{i=1}^l \hbar^i \sum_{\sigma(i)} e_{\alpha} \wedge e_{-\beta},$$

where $\sigma = \sum \sigma(i)$ is the subset of $\{(\alpha, \beta)\}$ that satisfies the following additional conditions: 1) The first entry (α) runs over a subalgebra Γ_1 of positive roots and the second entry (β) runs over a subalgebra Γ_2 of positive roots. 2) Both subalgebras are generated by simple roots and there is an isomorphism $\tau : \Gamma_1 \rightarrow \Gamma_2$, such that some iteration τ^k of τ leads out of Γ_1 ; $\tau^k \alpha \notin \Gamma_1$ for all $\alpha \in \Gamma_1$, and $(\alpha, \beta) \in \sigma$ iff $\beta = \tau^m \alpha$ for some positive integer m . 3) $w(\alpha) + w(\tau \alpha) = 0$. Finally, $(\alpha, \beta) \in \sigma(i)$ means that both roots have the same length i , and l is the rank of \mathfrak{g} .

?

3. Cohomology.

A. LIE ALGEBRAS.

Let \mathfrak{g} be a Lie algebra over \mathcal{C} , with structure tensor ϵ , and \mathfrak{g}^* its vector space dual. The cochains C_p^q are functions from $\mathfrak{g}^{\wedge p}$ to $\mathfrak{g}^{\wedge q}$, and the action of d on $\sigma \in C_p^q$ is given by

$$\begin{aligned} d\sigma(u_o, u_1, \dots, u_p) &= \sum_i (-)^i u_i \sigma(u_o, \dots, \hat{u}_i, \dots, u_p) \\ &+ \sum_{i < j} (-)^{i+j} \sigma([u_i, u_j], u_o, \dots, \hat{u}_i, \dots, \hat{u}_j, \dots, u_p) \end{aligned} \quad (3.1)$$

In the first term the element $u_i \in \mathfrak{g}$ acts on σ according by the adjoint action:

$$u\sigma := [\Delta_0^{p-1}u, \sigma] = [u, \sigma]. \quad (3.2)$$

The second formula should be taken to be short hand for the first one, where

$$\Delta_0 u = 1 \otimes u + u \otimes 1, \quad (3.3)$$

and Δ^p is the p 'th iterate of this primitive tensor product. In view of what happens later, it is worth notice that this notion of tensor product is needed from the beginning.

Let us examine the meaning of the simplest examples.

$p = 0$. If $\sigma \in C_0^q$, then

$$d\sigma(u) = [u, \sigma]. \quad (3.4)$$

Thus $d\sigma = 0$ means that σ is an **invariant element** of $\mathfrak{g}^{\wedge q}$, and the coboundaries $B_1^q \subset C_1^q$ are those cochains that can be expressed as a commutator; in particular, the **derived algebra** is $B_1^1(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$. The function $C = d\sigma$ from \mathfrak{g} to \mathfrak{g} generates an infinitesimal **inner homomorphism** of the Lie algebra,

$$u \mapsto u_\sigma = u + \hbar[\sigma, u]. \quad (3.5)$$

Consider the case $q = 2$; $r \in C_0^2$ is an element of $\mathfrak{g} \otimes \mathfrak{g}$. The solutions of $dr = 0$ are the invariant elements in $\mathfrak{g} \otimes \mathfrak{g}$, and $f = dr$ is a function from \mathfrak{g} to $\mathfrak{g} \wedge \mathfrak{g}$; this special case is central to our subject.

$p = 1$. If $C \in C_1^q$, then

$$dC(u, v) = [u, C(v)] - [v, C(u)] - C([u, v]). \quad (3.6)$$

If $q = 0$ then C takes its values in \mathcal{C} , the first two terms disappear, and $dC = 0$ means that C is an invariant \mathcal{C} -valued linear functional on \mathfrak{g} . If $q = 1$, then $C \in C_1^1$ is a function from \mathfrak{g} to \mathfrak{g} ; this finds application as a generator of a change of basis in the Lie algebra. Thus, consider the mapping

$$u \mapsto u_c = u + \hbar C(u); \quad (3.7)$$

then to order \hbar

$$[u_c, v_c] = [u, v] + \hbar([u, C(v)] - [v, C(u)]) = [u, v]_c + \hbar dC(u, v). \quad (3.8)$$

That is, C generates a homomorphism of \mathfrak{g} iff $dC = 0$. If $C = d\sigma$, then the homomorphism is an inner one, and $dC = 0$ is automatic since $d^2 = 0$. The functions dC are the **infinitesimal deformations** of the structure tensor induced by a reparameterization of the Lie algebra.

The case $q = 2$: let f be a function from \mathfrak{g} to $\mathfrak{g} \wedge \mathfrak{g}$, f can be interpreted as a **coproduct**,

$$\Delta_f(u) = f(u). \quad (3.9)$$

If $df = 0$, then this coproduct is compatible with the Lie structure of \mathfrak{g} and turns \mathfrak{g} into a **Lie bialgebra**, and if $f = dr$ we have a “coboundary Lie bialgebra”; we postpone further discussion of this point.

$p = 2$. When $E \in C_2^0$ we have $dE = \sum_{cyclic} E([u, v], w)$, and $dE = 0$ is the condition for E to define a **central extension** of \mathfrak{g} . The Lie structure tensor ϵ belongs to C_2^1 . For any such cochain our formula (3.1) gives

$$dE(u, v, w) = \sum_{cyclic} ([u, E(v, w)] + E([u, v], w)) \quad (3.10)$$

To understand the meaning of this condition consider a deformation of the structure

$$\epsilon \mapsto \epsilon' = \epsilon + \hbar E. \quad (3.11)$$

where \hbar is the deformation parameter. It is easy to see that $dE = 0$ is the condition that the deformed structure tensor satisfy the Jacobi identity, to first order in \hbar . If $E = dC$, then the deformation is induced by a change of basis and $dE = 0$ trivially. When the formula is applied to ϵ itself it yields

$$d\epsilon(u, v, w) = 2 \sum_{cyclic} \epsilon([u, v], w) = 2 \sum_{cyclic} \epsilon(\epsilon(u, v), w), \quad (3.12)$$

and $d\epsilon = 0$ is the Jacobi identity. Thus (3.10) is the linearization of (3.12).

Let $Z_p^q \subset C_p^q$ be the closed cocycles, $B_p^q \subset Z_p^q$ the coboundaries, and $H_p^q = Z_p^q/B_p^q$. We have seen that H_0^q consists of the invariants in $\mathfrak{g}^{\wedge q}$, H_1^0 is the set of invariant functionals on \mathfrak{g} , H_1^1 is the set of outer automorphisms of \mathfrak{g} , H_1^2 classifies the non-coboundary Lie bialgebras and H_2^1 contains the essential deformations of the structure. The obstructions to carrying the deformations to higher order lie in H_3^1 , and H_2^0 classifies the central extensions of \mathfrak{g} . The Haar measure (on the group) belongs to H_n^0 , $n = \dim(\mathfrak{g})$.

If \mathfrak{g} is **simple**, then $H_p^q = 0$ whenever $q \neq 0$; H_1^0, H_2^0 also vanish, and $H_3^0 \neq 0$. In particular, $H_1^2 = 0$ means that all structures of Lie bialgebra are of the coboundary type, $H_2^1 = 0$ means that the Lie structure is **rigid**, and $H_2^0 = 0$ that there are no central extensions.

Let $S \subset T$ be \mathfrak{g} -modules and Q the \mathfrak{g} -module T/S ; then T is called an **extension** of S by Q . As vector spaces, $T = Q \oplus S$; let $\pi_0 = \pi_Q \oplus \pi_S$, then there is a map C from \mathfrak{g} to $\text{Hom}(Q, S)$ such that

$$\pi_T(u) = \pi_0(u) + C(u),$$

where for $u \in \mathfrak{g}$,

$$C(u) : q \oplus s \mapsto 0 \oplus C(u)q. \quad (3.13)$$

The fact that π_T is a representation of \mathfrak{g} amounts to $dC = 0$,

$$[\pi_T(u), \pi_T(v)] - \pi_T([u, v]) = [u, C(v)] - [v, C(u)] - C([u, v]) = dC(u, v).$$

Suppose there is a map $\sigma : Q \rightarrow S$ such that

$$C(u) = [\pi_0(u), \sigma] = \pi_S(u)\sigma - \sigma\pi_Q(u) = d\sigma(u); \quad (3.14)$$

then $dC = 0$. We show that in this case the extension is trivial (split). Change basis by setting

$$q \oplus s' = q \oplus (s + \sigma q),$$

then using (3.14) one obtains

$$\pi_T : q \oplus s' \mapsto \pi_Q q \oplus \pi_S s'.$$

Conclusion: if $C = d\sigma$, then π_T is equivalent to π_0 . This example of cohomology is fundamental; some of the examples discussed above can be interpreted this way, with proper identification of the spaces.

B. LIE BIALGEBRAS.

Suppose now that \mathfrak{g}^* also has a Lie structure, with structure tensor f . By pullback, or duality, the cochains that were seen as functions from $\mathfrak{g}^{\wedge p}$ to $\mathfrak{g}^{\wedge q}$, can be interpreted as functions from $\mathfrak{g}^{*\wedge q}$ to $\mathfrak{g}^{*\wedge p}$, or more symmetrically, from $\mathfrak{g}^{\wedge p} \otimes \mathfrak{g}^{*\wedge q}$ to \mathbb{C} . (In this last case it is natural to replace \mathbb{C} by any space that is both a \mathfrak{g} -module and a \mathfrak{g}^* -module.) Now we have two differential operators,

$$d : C_p^q \rightarrow C_{p+1}^q, \quad \partial : C_p^q \rightarrow C_p^{q+1}.$$

This is what is called a **bi-complex**, provided that d, ∂ anticommute: $d \circ \partial + \partial \circ d = 0$. The structure is defined by $\epsilon \in C_2^1$ and $f \in C_1^2$; the Jacobi identities are $d\epsilon = 0$, $\partial f = 0$.

Theorem. The following statements are equivalent

$$1) \ d\partial + \partial d = 0, \ 2) \ df = 0, \ 3) \ \partial\epsilon = 0.$$

Strictly speaking, an overall sign has to be adjusted in the definition of ∂ to get the operators to anticommute. The equivalence of 2) and 3) is easy to prove. On the dual bases $(L_i, \Gamma^i)_{i=1, \dots, \dim(\mathfrak{g})}$,

$$(df)_{ij}^{kl} = \epsilon_{im}^k f_j^{ml} + \epsilon_{im}^l f_j^{km} - (i, j) - \epsilon_{ij}^m f_m^{kl} = (\partial\epsilon)_{ij}^{kl}. \quad (3.17)$$

The last equality (up to a sign, perhaps) is an evident consequence of the symmetry of the tensor with respect to the roles played in it by the two structure tensors.

When $df = 0$ we say that the structures f and ϵ are **compatible**. The theorem gives a first interpretation of what this means. From now on we are given a pair of compatible forms f and ϵ .

From the discussion of the connection between $d\epsilon = 0$ and the Jacobi identity it follows that the form f is ∂ -closed, so

$$\partial f = 0, \quad df = 0.$$

The first equation says that f defines a Lie structure on \mathfrak{g}^* and the second that it is compatible with the Lie structure of \mathfrak{g} .

Alternatively, one can ascribe both structures to \mathfrak{g} , associating to f the coproduct $\Delta_f: \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$,

$$\Delta_f(u) = f(u).$$

The subscript is needed to distinguish Δ_f from the primitive coproduct Δ_0 , Eq.(3.3). This double structure makes \mathfrak{g} into a Lie bialgebra:

$$[u, v] = \epsilon(u, v), \quad \Delta_f(u) = f(u), \quad \partial f = df = 0 = \partial\epsilon = d\epsilon.$$

In particular, if f is a coboundary,

$$f = dr, \quad r \in C_0^2,$$

then $df = 0$ automatically, while $\partial f = 0$ becomes a condition on r ,

$$\partial f = \partial dr = -d\partial r = 0. \tag{3.18}$$

That is, ∂r must be an invariant element of $\mathfrak{g}^{\wedge 3}$. The formula is

$$\partial r = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] =: \text{CYB}(r), \tag{3.19}$$

and $\partial r = 0$ is the classical Yang-Baxter relation.

Strictly, this makes sense if r is antisymmetric. It turns out that, by including in r a symmetric, invariant tensor, one can replace $d\partial r = 0$ by the more convenient condition $\partial r = 0$. From now we assume, following Drinfeld, that

$$r + r^t = K, \quad \partial r = 0. \tag{3.20}$$

Here K is the Killing form of \mathfrak{g} and ∂r is defined by (3.19).

The dual, \mathfrak{g}^* , also becomes a Lie bialgebra, with

$$\{a, b\} = f(a, b), \quad \Delta_\epsilon a = \epsilon(a).$$

It is natural to ask if this is also of the boundary type.

Let A, U be the matrices of the adjoint actions of $a \in \mathfrak{g}^*$ and $u \in \mathfrak{g}$,

$$A_j^i = a_k f_j^{ik}, \quad U_j^i = u^k \epsilon_{jk}^i. \quad (3.21)$$

The following are identities:

$$\begin{aligned} rU + Ur + f(u) &= 0, \\ rA + Ar + r\epsilon(a)r &= 0. \end{aligned} \quad (3.22)$$

The first is just a way to write $f(u) = [u, r]$, or $f = dr$; the second is the same as the classical Yang-Baxter relation, $\partial r = 0$. If r , considered as a map from \mathfrak{g} to \mathfrak{g}^* , were invertible, then it would follow that $\epsilon(a) = [a, r^{-1}]$, or $\epsilon = \partial r^{-1}$. In fact, r is not invertible, nevertheless we shall see that

NB

C. MANIN ALGEBRA.

Another implication of compatibility of f, ϵ is the existence of a structure of Lie algebra on $\mathfrak{g} \oplus \mathfrak{g}^*$. Given the brackets on \mathfrak{g} and on \mathfrak{g}^* one adds ($a \in \mathfrak{g}^*$ and $u \in \mathfrak{g}$)

$$[a, u] = \text{coad}_a u - \text{coad}_u a,$$

where

$$\text{coad}_a u(b) = -u(\{a, b\}), \quad \text{coad}_u a(v) = -a([u, v]).$$

The Jacobi identity is satisfied by virtue of $df = 0$. The Lie algebra obtained in this way is denoted $\mathfrak{g} \bowtie \mathfrak{g}^*$.

Theorem. The Manin algebra $\mathfrak{g} \bowtie \mathfrak{g}^*$ is isomorphic to $\mathfrak{g} \oplus \mathfrak{g}$.

We give a constructive proof.

Consider the adjoint action of $\mathfrak{g} \bowtie \mathfrak{g}^*$ on itself. Let $u, v \in \mathfrak{g}$ and $a, b \in \mathfrak{g}^*$; then

$$ad_{(u,a)}(v, b) = (vA - f(u)b - vU, Ub - Ab - \epsilon(a)v). \quad (3.23)$$

with A, U as in (3.21). The action of \mathfrak{g} involves an extension of the coadjoint representation by the adjoint representation. Since $f = dr$, we can trivialize this extension by a change of variables. Define $v' = v - rb$, then

$$ad_{(u,0)}(v', b) = (-v'U, Ub), \quad (3.24)$$

by virtue of the first of Eq.s (3.22). The action $ad_{(0,a)}$ likewise involves an extension, but since \mathfrak{g}^* is not simple it is not clear whether it is trivial.

The problem is to determine the invariant subspaces for the action in (3.23). By (3.24) every subspace that is invariant under $ad_{(u,0)}$ has the form

$$S_\kappa = \{(v, b); v' = \kappa Kb\},$$

with $\kappa \in \mathcal{C}$ and K the Killing form. This subspace is invariant under $ad_{(0,a)}$ iff

$$A(\kappa K - r) + (\kappa K - r)A - (\kappa K - r)\epsilon(a)(\kappa K - r) = 0.$$

When $\kappa = 0$ this reduces to the second of Eq.s(3.22), which is the classical Yang-Baxter relation; the latter is also obeyed by $r^t = K - r$, so the statement is valid for $\kappa = 1$ as well. That is; S_0, S_1 are invariant subspaces of the adjoint action of $\mathfrak{g} \bowtie \mathfrak{g}^*$. The actions are determined by

$$ad_{(u,a)} : b \mapsto \begin{cases} (U - A - \epsilon(a)r)b, & (v, b) \in S_0, \\ (U - A + \epsilon(a)r^t)b, & (v, b) \in S_1. \end{cases}$$

The two subalgebras commute, $[S_0, S_1] = 0$; $(ra, a) \in S_0$ and $(r^t a, -a) \in S_1$ act by the matrix $-\text{ad}_{Ka}$ on S_0 and on S_1 , respectively. The theorem is proved, $\mathfrak{g} \bowtie \mathfrak{g}^*$ is isomorphic to $S_0 \oplus S_1 = \mathfrak{g} \oplus \mathfrak{g}$. This result was first obtained by Reshetikhin and Semenov-Tian-Shanskii [14].

Consider now the bicomplex

$$\begin{array}{ccccccc}
C_0^0 & \xrightarrow{d} & C_1^0 & \xrightarrow{d} & C_2^0 & \xrightarrow{d} & C_3^0 \quad \dots \\
\partial \downarrow & & \partial \downarrow & & \partial \downarrow & & \partial \downarrow \\
C_0^1 & \xrightarrow{d} & C_1^1 & \xrightarrow{d} & C_2^1 & \xrightarrow{d} & C_3^1 \quad \dots \\
\partial \downarrow & & \partial \downarrow & & \partial \downarrow & & \partial \downarrow \\
C_0^2 & \xrightarrow{d} & C_1^2 & \xrightarrow{d} & C_2^2 & \xrightarrow{d} & C_3^2 \quad \dots \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

From the perspective of $\mathfrak{g} \bowtie \mathfrak{g}^*$, \mathfrak{g} and \mathfrak{g}^* are subalgebras, and the chains are

$$C^{i-2} = C_0^i \cup C_1^{i-1} \cup \dots \cup C_{i-1}^1 \cup C_i^0,$$

functions from $(\mathfrak{g} \bowtie \mathfrak{g}^*)^{\wedge n}$ to \mathbb{C} (or to some $\mathfrak{g} \bowtie \mathfrak{g}^*$ -module M). Thus it is natural to consider the chain $C^1 = C_1^2 \cup C_2^1$, where we find the tensors f and ϵ united, both being interpreted as functions from $\mathfrak{g} \bowtie \mathfrak{g}^* \otimes \mathfrak{g} \bowtie \mathfrak{g}^*$ to $\mathfrak{g} \bowtie \mathfrak{g}^*$. It is customary to start the complex at $p, q = 1, 1$; that explains the strange notation. The cohomology space associated with C^1 is called H^1 or more precisely $H^1((\mathfrak{g} \bowtie \mathfrak{g}^*)^{\wedge 2})$ since $H^1(M)$ normally is taken to mean the homology group with values in the module M .

The complex built on these chains is called the total complex. The differential operator of the total complex is $d + \partial$. It is natural to use this complex to investigate the deformations of $\mathfrak{g} \bowtie \mathfrak{g}^*$; that is, simultaneous deformations of f and ϵ . Since \mathfrak{g} is simple, $\mathfrak{g} \bowtie \mathfrak{g}^*$ is semisimple and $H^1(M) = 0$ for all M . In particular, that $H^1((\mathfrak{g} \bowtie \mathfrak{g}^*)^{\wedge 2})$ is trivial means that $\mathfrak{g} \bowtie \mathfrak{g}^*$ is rigid, all Lie algebra deformations are trivial. Of course, this does not mean that the deformations are uninteresting. We saw that the classical Yang-Baxter relation means that $\partial r = 0$, while $dr = f$ does not vanish; the classification of r -matrices is not a problem of cohomology on the Manin algebra.

4. Quantization.

A. SYMPLECTIC STRUCTURE AND QUANTIZATION.

Let \mathfrak{g} be a simple, complex Lie algebra, \mathfrak{g}^* its vector space dual, G the associated local Lie group and $A_0 = A_0(G)$ the germ of functions at the identity. Whenever convenient we refer to dual bases (L_i, Γ^i) $i = 1, 2, \dots, n$ for \mathfrak{g} and \mathfrak{g}^* . The Γ^i may be interpreted as a set of exponential coordinates on G ; for $g \in G$ one has $g = e^{\Gamma(g)}$, $\Gamma(g) = \Gamma^i(g)L_i$. The structure tensor for \mathfrak{g} is denoted ϵ , $[L_i, L_j] = \epsilon_{ij}^k L_k$.

The view of quantization presented here is the same as that of ref [11], with the new elements introduced by Drinfeld [2]. One regards G as a phase space and $A_0(G)$ as the space of classical observables. We suppose that A_0 is endowed with a Poisson bracket $\{, \}$, determined by a Poisson form f ,

$$\{\Gamma^i, \Gamma^j\} = f_k^{ij} \Gamma^k, \quad (4.1)$$

The coefficients are antisymmetric in i, j and satisfy the Jacobi identity,

$$\sum_{(ijk)} f_n^{im} f_m^{jk} = 0. \quad (4.2)$$

It is possible to identify phase space with a symplectic leaf of G , rather than G itself. The space $A_0(G)$ is thus furnished with two independent structures; the commutative algebra of functions,

$$(ab)(g) = a(g)b(g), \quad a, b \in A_0, \quad g \in G, \quad (4.3)$$

and the Poisson-Lie bracket [2]

$$\{a, b\} = \Lambda^{ij} [(L_i a)(L_j b) - (L'_i a)(L'_j b)], \quad \Lambda^{ij} = \Gamma^k f_k^{ij}, \quad (4.4)$$

where $L_i(L'_i)$ is to be understood as a vector field of left (right) translations.

The aim of quantization is to replace both structures by a single, noncommutative product denoted $a * b$, $a, b \in A_0(G)$, expressed as a formal deformation of the Poisson bracket,

$$a * b = \sum_{n=0}^{\infty} \hbar^n C_n(a, b) = ab + (i\hbar/2)\{a, b\} + o(\hbar^2), \quad (4.5)$$

with deformation parameter \hbar (Planck's constant).

We insist on associativity of the new product, and this implies conditions on the C_n 's that are best formulated in terms of the Hochschild cohomology of A_0 .

The Hochschild cochains are in this instance linear functions from $A^{\otimes n}$ to A and

$$\begin{aligned} dC(a_1, \dots, a_{p+1}) &= a_1 C(a_2, \dots, a_{p+1}) \\ &+ \sum (-)^i C(a_1, \dots, a_i a_{i+1}, \dots, a_{p+1}) \\ &+ (-)^{p+1} C(a_1, \dots, a_p) a_{p+1}. \end{aligned}$$

The associativity constraint is

$$\begin{aligned} a * (b * c) - (a * b) * c &= \sum_0^\infty \hbar^n D_n(a, b, c) = 0, \\ D_n &= E_n - dC_n \in C_3^1, \\ E_n(a, b, c) &= \sum_{r=1}^{n-1} C_r(C_{n-r}(a, b), c) - C_r(a, C_{n-r}(b, c)). \end{aligned}$$

One can verify that, if $D_n = 0$ for $n \leq m$ then $dE_{m+1} = 0$. To continue the deformation to the next order one must find C_{m+1} such that $E_{m+1} = dC_{m+1}$. Therefore E_{m+1} must be in the zero class of H_3^1 (usual name $H^3(A)$). One can also show that two $*$ -products are equivalent if the difference of their cochains are exact at each level; inequivalent $*$ -products are therefore classified by $H_2^1 (= H^2(A))$.

The odd part $a * b - b * a$ of the product defines a structure of Lie algebra. The Jacobi identity implies conditions on the C_n 's that are expressed in terms of the Chevalley-Eilenberg cohomology of $\{, \}$ and are automatically satisfied by associativity of $*$.

We denote by $A_\hbar(G)$ the algebra of formal series in \hbar with coefficients in A_0 , with an associative $*$ -product.

B. $*$ -PRODUCTS AND ABSTRACT ALGEBRAS.

Quantization as exposed so far is characterized by a well defined correspondence between the quantum algebra A_\hbar and the classical algebra A_0 ; every element in A_\hbar is identified with a power series in \hbar with coefficients in A_0 and, when the series converges, with a unique function on phase space. The advantage of this approach is considerable [11],[12]; for example, integration on the quantum algebra is readily available.

A more abstract point of view is to study the abstract, unital algebra generated by elements $(\Gamma_i)_{i=1,2,\dots,n}$, with relations such as

$$\Gamma^i * \Gamma^j - \Gamma^j * \Gamma^i = i\hbar f_k^{ij} \Gamma^k. \quad (4.6)$$

Apart from the fact that this may be a proper subalgebra of A_\hbar (which is not the issue I want to discuss) I point out that one here abandons the actual interpretation of an element of the algebra A as a function on G , making the term “quantum algebra of functions” somewhat misleading. (Relations of the type (4.6) do not necessarily hold, the point is that when they do, then the alternative, abstract point of view becomes available. This may also be possible with other types of relations, as with certain types of quadratic relations (Sklyanin algebras), but a detailed investigation is required to determine the nature of an abstract algebra defined by relations not of the Lie type.)

C. COMPATIBILITY.

We have made no use, so far, of the group structure on our phase space. It gives rise to a condition of “compatibility” between the Poisson structure on \mathfrak{g}^* and the Lie bracket on \mathfrak{g} , namely $df = 0$, or in components

$$[L_i, f_j] - [L_j, f_i] - \epsilon_{ij}^k f_k = 0, \quad f_k := f_k^{mn} L_m \otimes L_n. \quad (4.7)$$

The real meaning of this relation must be clearly grasped; we begin by eliminating some potential misconceptions. See Drinfeld, ref. [2].

The Lie structure of \mathfrak{g}^* gives rise to a coproduct on \mathfrak{g} , $\Delta_f : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$, defined by

$$\Delta_f(L_k) = f_k. \quad (4.8)$$

This is a standard application of duality,

$$(\Delta_f L_k)(\Gamma^m, \Gamma^n) = L_k(\{\Gamma^m, \Gamma^n\}).$$

Satisfying the condition of “compatibility” does not turn Δ_f into a homomorphism, neither does Δ_f become an intertwiner the adjoint action, though this misses by a factor of two.

A closer look at (4.7) reveals that it involves the standard coproduct on the enveloping algebre $U(\mathfrak{g})$, generated by

$$\Delta_0 : L_i \mapsto L_i \otimes 1 + 1 \otimes L_i, \quad (4.9)$$

since the first two terms in (4.7) involves the adjoint action of \mathfrak{g} on $\mathfrak{g} \otimes \mathfrak{g}$. So the real meaning of the compatibility condition should be sought in the context of the enveloping algebra.

Indeed, the condition of compatibility, Eq(4.7), is the condition that there is a homomorphism to order \hbar , ($U = U(\mathfrak{g})$ from now on)

$$\Delta_{\hbar} : U \rightarrow U \otimes U$$

generated by

$$\Delta_{\hbar}(L_i) = L_i \otimes 1 + 1 \otimes L_i + (i\hbar/2) f_i^{mn} L_m \otimes L_n; \quad (4.10)$$

that is,

$$\Delta_{\hbar}([L_i, L_j]) = [\Delta_{\hbar}(L_i), \Delta_{\hbar}(L_j)] + o(\hbar^2).$$

This last coincides with (4.7) and expresses the true meaning of the compatibility condition. This interpretation is dual to Drinfeld's Poisson Lie groups [2].

This establishes a firm link between the compatibility condition, with its well known connection to bialgebra structure and cohomology, and the type of deformation that characterizes quantization. The condition is the new element that comes into play when phase space is a group manifold.

D. FURTHER COMMENTS ON QUANTIZATION.

Let a $*$ -product be given on $A_0(G)$ and suppose for concreteness that relations (4.7),

$$\Gamma^i * \Gamma^j - \Gamma^j * \Gamma^i = i\hbar f_k^{ij} \Gamma^k. \quad (4.11)$$

hold exactly. As we said, we may choose to forget the origin of this relation and use it only to define an abstract algebra generated by elements $(\Gamma^i)_{i=1,2,\dots,n}$. Now conversely, given this abstract algebra, we may try to realize it concretely in terms of a $*$ -product on $A(G)$. This amounts to establishing (choosing) a mapping from the

abstract algebra into A_0 ; such a map is called an “ordering”. The most traditional way to do this is to set up a basis for the abstract algebra, for example

$$(\Gamma^1 *)^{k_1} \dots (\Gamma^n *)^{k_n} =: \Gamma_*^{\{k\}},$$

and a linear correspondance that associates each such to a specific function on G , for example, to the function

$$(\Gamma^1)^{k_1} \dots (\Gamma^n)^{k_n} =: \Gamma^{\{k\}}, \quad (4.12)$$

Normal ordering and standard ordering are both of this type. The same method can be applied when the relations are of quadratic type.

Our favorite method is a generalization of symmetric ordering. Let

$$e_*^{\Gamma \cdot L / i\hbar} := \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{1}{i\hbar} \Gamma \cdot L * \right)^n, \quad \Gamma \cdot L := \Gamma^i L_i; \quad (4.13)$$

where the L_i are interpreted as coordinates on \mathfrak{g}^* and on G^* , the formal group generated by \mathfrak{g}^* . To this expression, interpreted as a formal series, one associates a specific function $E_*(L_1, \dots, L_n)$ on G , for example the function

$$e^{\Gamma \cdot L / i\hbar} := \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{1}{i\hbar} \Gamma \cdot L \right)^n.$$

It should be emphasized that the choice that is made for this function E_* (the “star-exponential”) is far from being without consequence; the image of the ordering map may even be finite dimensional for certain choices. See ref. [15].

The expression (4.13) can also be interpreted as an element of the local group G^* with Lie algebra \mathfrak{g}^* and structure tensor f . We have

$$e_*^{\Gamma \cdot L / i\hbar} \otimes e_*^{\Gamma \cdot L / i\hbar} = e_*^{\Gamma \cdot C(L) / i\hbar}, \quad (4.14)$$

where

$$C(L_i) = L_i \otimes 1 + 1 \otimes L_i + (i\hbar/2) f_i^{mn} L_m \otimes L_n + 0(\hbar^2); \quad (4.15)$$

This is just the Campbell-Hausdorff formula; in it $L_i \otimes L_j$ stands for coordinates for $\mathfrak{g}^* \otimes \mathfrak{g}^*$. To order \hbar , $C(L)$ coincides with $\Delta_{\hbar}(L)$, so the following conjecture

$$e_*^{\Gamma \cdot L / i\hbar} \otimes e_*^{\Gamma \cdot L / i\hbar} = e_*^{\Gamma \cdot \Delta_{\hbar}(L) / i\hbar}, \quad (4.16)$$

yields Δ_{\hbar} correctly to order \hbar . In fact, if \mathfrak{g} were abelian, then this formula would define a coproduct on U , compatible with the (trivial) Lie structure of \mathfrak{g} .

We can reverse the roles of \mathfrak{g} and \mathfrak{g}^* (of ϵ and f), since the compatibility condition is symmetric, and consider

$$e_*^{\Gamma \cdot L / i\hbar} \otimes e_*^{\Gamma \cdot L / i\hbar} = e_*^{\Delta_{\hbar}(\Gamma) \cdot L / i\hbar}, \quad (4.17)$$

with

$$\Delta_{\hbar}(\Gamma^k) = \Gamma^k \otimes 1 + 1 \otimes \Gamma^k + (1/i\hbar) \epsilon_{ij}^k \Gamma^i \otimes \Gamma^j. \quad (4.18)$$

This brings to mind the universal T-matrix [9],[10].

E. THE OPERATORS d AND ∂ ON U_{\hbar} .

Denote by U_{\hbar} the bialgebra dual of $A_{\hbar}(g^*)$, endowed with a (possibly deformed) algebraic structure and the coproduct Δ_{\hbar} as in Section 4D. Compatibility is generalized or lifted up in the most natural way from (4.9) to

$$\Delta_{\hbar}(uv) = \Delta_{\hbar}(u) \Delta_{\hbar}(v), \quad u, v \in U_g.$$

We continue to use a $*$ for the product on A_{\hbar} , whenever that serves to facilitate the interpretation of the formulas, but not for the product on U_{\hbar} , and write U, A and Δ without the subscript from now on. Unless explicit exception is made, all that follows is independent of the (possibly restrictive) condition that Δ be expressible in the form (4.18).

The basic cohomology on A is Hochschild, with cochains

$$C_p^q \in \text{Hom}(A^{\otimes p}, A^{\otimes q}),$$

The formula for the differential operator is

$$\begin{aligned} dC(a_1, \dots, a_{p+1}) &= \Delta^{q-1}(a_1)C(a_2, \dots, a_{p+1}) \\ &\quad + \sum (-)^i C(a_1, \dots, a_i a_{i+1}, \dots, a_{p+1}) \\ &\quad + (-)^{p+1} C(a_1, \dots, a_p) \Delta^{q-1}(a_{p+1}). \end{aligned} \quad (4.19)$$

with $a_1, \dots, a_{p+1} \in A$, $C(a_1, \dots, a_p) \in A^{\otimes q}$. To get the flavor of it, consider the case $p = 0, q = 1$, so that $C \in A$, then from (4.19), the first and the last terms contributing,

$$(dC)(a) = aC - Ca;$$

and (4.19) can be considered as a generalization of this formula, in two steps. The generalization to the case where C is in $A^{\otimes q}$ needs an action of $a \in A$ on this object; it is given by $\Delta^{q-1}(a)$. The generalization to higher forms follows the pattern set by the de Rham complex.

The dual U of A is an algebra, and it has its own Hochschild complex with differential operator ∂ . By duality, this operator can be expressed in terms of the coproduct of A . While d increases the number of arguments, leaving the value in the same space, ∂ maps the function taking values in $A^{\otimes q}$ to one taking values in $A^{\otimes q+1}$. To write it we need some notation. The coproduct can be expressed as a sum,

$$\Delta(a) = \sum_{(a)} a^1 \otimes a^2,$$

and this will be the meaning of a^1, a^2 in what follows. The formula is

$$\begin{aligned} \partial C(a_1, \dots, a_p) &= \sum_{(a_1, \dots, a_p)} a_1^1 a_2^1 \dots a_p^1 \otimes C(a_1^2, \dots, a_p^2) \\ &\quad + \sum_i (-)^i \Delta_i C(a_1, \dots, a_p) \\ &\quad + (-)^{q+1} \sum_{(a_1, \dots, a_p)} C(a_1^1, \dots, a_p^1) \otimes a_1^2 \dots a_p^2. \end{aligned} \tag{4.20}$$

This can be considered as a generalization of $\Delta : U \rightarrow U \otimes U$, when $p = 0, q = 1, C \in U$ and $dC = \Delta C$. To illustrate the formula, consider the first term in the case that $p = 1$, then for $u \in U$,

$$(uC)(a) = (u \otimes C)(\Delta a) = \sum_{(a)} u(a^1) C(a^2). \tag{4.21}$$

Let us indicate the values $C(a; u_1, \dots, u_q)$ of $C(a)$. The last formula becomes

$$\begin{aligned} (uC)(a; u_1, \dots, u_q) &= (u \otimes C(u_1, \dots, u_q))(\Delta a) = \sum_{(a)} u(a^1) C(a^2; u_1, \dots, u_q) \\ &= \left(\sum_{(a)} a^1 \otimes C(a^2) \right) (u \otimes u_1 \otimes \dots \otimes u_q). \end{aligned}$$

We see that it is the value of a one-form on A ; it is the value of the first contribution to $\partial C(a)$ at arguments u, u_1, \dots, u_q .

Let us go a step further, to the case $p = 2$, by representing $C(a_1, a_2)$ as $(\Delta D)(a_1, a_2) = D(a_1 * a_2)$. (This is sufficient, by linearity.) Then one gets an action of $u \in U$, from $\text{Hom}(A^{\otimes 2}, A^{\otimes q})$ to the same with q replaced by $q + 1$,

$$(uC)(a, b; u_1, \dots, u_q) = \sum_{(a)(b)} \Delta(u)(a^1, b^1) \otimes C(a^2, b^2; u_1, \dots, u_q).$$

In general

$$(uC)(a_1, \dots, a_p)(u_1, \dots, u_q) = (\Delta^{p-1}(u) \otimes C(u_1, \dots, u_q)(\Delta a_1, \dots, \Delta a_p)), \quad (4.22)$$

with the understanding that $\Delta^{p-1}(u)$ takes the arguments a_1^1, \dots, a_p^1 and $C(u_1, \dots, u_q)$ the arguments a_1^2, \dots, a_p^2 . This is the value of the first term in Eq(4.20) at u, u_1, \dots, u_q .

Checking the properties $d^2 = 0$ and $\partial^2 = 0$, we find that they are equivalent to

$$\Delta(a)\Delta(b) = \Delta(a * b), \text{ and } (1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta,$$

respectively.

As a first application, let us take $p = q = 1$, and write the definitions again,

$$dC(a_1, a_2; b) = (a_1 C(a_2))(b) - C(a_1 a_2)(b) + (C(a_1) a_2)(b),$$

$$\partial C(a; b_1, b_2) = \sum_{(a)} (a^1 \otimes C(a^2))(b_1, b_2) - (\Delta C(a))(b_1, b_2) + \sum_{(a)} (C(a^1) \otimes a^2)(b_1, b_2),$$

The most interesting cochains are $UT \in C_1^1$, defined by $UT(a) = a$, the multiplication map $m \in C_2^1$, and the coproduct $\Delta \in C_1^2$:

$$dUT = m, \quad \partial UT = \Delta. \quad (4.23)$$

It follows that $dm = 0 = \partial \Delta$, and indeed

$$\begin{aligned} dm(a, b, c) &= 2(a * (b * c - (a * b) * c)), \\ \partial \Delta &= 2((1 \otimes \Delta)\Delta - (\Delta \otimes 1)\Delta). \end{aligned} \quad (4.24)$$

In the case of a Hopf algebra the properties of the counit and the antipode can also be expressed neatly in terms of the differential operators.

Exercise. When the operator d is applied to $C_p^0 = U^{\otimes p}$, it yields a map from $U^{\otimes p}$ to $U^{\otimes p+1}$. So in this case d is represented by an element $C \in C_p^{p+1}(U)$. Investigate the question of whether these cochains are closed and/or coclosed.

F. THE R-MATRIX.

In this section let Δ denote the coproduct on U and Δ' the opposite coproduct,

$$\Delta(u) = \sum_{(u)} u^1 \otimes u^2, \quad \Delta'(u) = \sum_{(u)} u^2 \otimes u^1.$$

Suppose further that there is an element $R \in U \otimes U$ such that

$$\Delta'(u)R = R\Delta(u). \quad (4.25)$$

We have $\Delta' = \sigma\Delta$, where σ interchanges the factors, so if we define $P \in U \otimes U$ by

$$P = \sigma R,$$

then Eq(4.25) becomes

$$\Delta(u)P - P\Delta(u) = 0, \quad \text{or} \quad \partial P = 0. \quad (4.26)$$

Remember that ∂ is the “ d ” of the algebra U , applied to our complex via the identification $u(a) = a(u)$. Thus to apply the formula (4.20) for ∂ one should interpret P as a function from $A \otimes A$ to \mathbb{C} . Then ∂P is a function from $A \otimes A$ to A , which attests to its having to do with multiplication on A . By these conventions we avoid having to deal with two different double complexes, each with two differentiations.

The matrix R is said to be “unitary” if $R_{12}R_{21} = 1$. We have $\sigma R_{12}R_{21} = P^2$, and evidently $dP^2 = 0$. This points at a hierarchy of operators,

$$I_1 = R, \quad I_2 = R_{12}R_{21}, \quad I_3 = R_{12}R_{13}R_{23}/R_{23}R_{13}R_{12}, \dots$$

In terms of them we can express triviality, $I_1 = 1$, unitarity, $I_2 = 1$ and the Yang-Baxter relation, $I_3 = 1$. Multiplying by σ one turns these operators into the invariants

$$J_1 = P, \quad J_2 = P^2, \quad J_3 = (P_{12}P_{23}P_{12}/P_{23}P_{12}P_{23}).$$

We have seen that $\partial P = 0$ and thus $\partial P^2 = 0$ as a direct consequence of (4.24). As to the third invariant there is a well known algorithm using associativity of the product on A that leads to $\partial J_3 = 0$. This last is weaker than the braid relation, which makes $J = 1$ and which is equivalent to the Yang-Baxter relation. I am unaware of any convincing

argument that justifies YB on the basis of associativity alone. The statement that $\partial J = 0$ is trivial since $\partial P = 0$; the fact that it can be derived using associativity is fortuitous.

Before going on I want to rephrase the theory in the Woronowicz picture. The local group G (with Lie algebra \mathfrak{g}) is contained in the (undeformed) enveloping algebra U_0 of \mathfrak{g} . For $g \in G$ the standard (classical) coproduct takes the form

$$\Delta_0(g) = g \otimes g. \quad (4.27)$$

Assume that there is a subset G of the deformed algebra U that preserves the structure of the group, and that the group algebra is dense in U . (To simplify, we can even assume, with no essential loss of generality [13], that the algebraic structure of U is preserved; this is what is called a preferred deformation [16].) Let π be a fundamental representation of G , of dimension n , and extend it to U .

Following Woronowicz [6], define the functions $(T_i^j)_i, j = 1, \dots, n$ on U by

$$T_i^j(g) = (\pi g)_i^j, \quad g \in G. \quad (4.28)$$

The coproduct on A is generated by

$$\begin{aligned} (\Delta T)_i^j(g_1 \otimes g_2) &= (\pi g_1 g_2)_i^j = \pi(g_1)_i^k \pi(g_2)_k^j, \\ \Delta T_i^j &= T_i^k \otimes T_k^j. \end{aligned}$$

Another application of duality gives

$$(T_i^j * T_k^l)(g) = \Delta(\pi g)_{ik}^{jl}, \quad (4.29)$$

where if $\Delta(g) = \sum_{(u)} u^1 \otimes u^2$, then $\Delta(\pi g) = \sum_{(u)} \pi(u^1) \otimes \pi(u^2)$. Now evaluate Eq(4.26) in π , replacing u by g :

$$\Delta(\pi g)P - P\Delta(\pi g) = 0,$$

where P now should be understood as the matrix πP . In view of (4.29) this can be read as

$$T_i^j * T_k^l P_{jl}^{mn} - P_{ik}^{jl} T_j^m * T_l^n = 0, \quad \text{or} \quad [T * T, P] = 0. \quad (4.30)$$

We have thus learned that these relations, that define the structure of the algebra generated by the T_i^j , is one way to write $\partial P = 0$.

The relations (4.29) have the form $Q = 0$, where Q is a fixed mapping from U to $U \otimes U$, or from $A \otimes A$ to A . Since $Q = \partial P$ it is evident that $\partial Q = 0$. More generally, suppose that the relations of A have the form $Q = 0$, with Q an element of C_2^1 . What do we buy with the restriction $\partial Q = 0$? Interpreting Q as an element of C_2^1 we have

$$(\partial Q)(a, b) = \sum_{(a)(b)} a^1 * b^1 \otimes Q(a^2, b^2) - \Delta Q(a, b) + \sum_{(a)(b)} Q(a^1, b^1) \otimes a^2 * b^2.$$

Evaluate this two-form at $(u, v) \in U \otimes U$. Assuming that $Q(a, b)$ vanishes at u and at v , we obtain

$$\Delta Q(a, b)(u, v) = Q(a, b)(uv) = 0;$$

that is, the relations are compatible with the structure of U . In other words, if T_1, T_2 are two sets of generators, of commuting copies of A , then the matrix elements of the matrix product $T_1 T_2$ satisfies the relations of A . This is just what makes A into a bialgebra; in the application to physical models, where T is the transition matrix, it is the key to integrability.

Conclusion: If the relations of a bialgebra A are given in the form $Q = 0$, Q an element of C_2^1 , then $\partial Q = 0$; among such bialgebras those characterized by an R-matrix are precisely those for which Q is exact. This justifies the terminology of Drinfeld, who calls them coboundary bialgebras. To be precise, one should insist that all the relations that define A are contained in the statement $Q = 0$; the restriction of quantum $gl(n)$ to quantum $sl(n)$ by fixing the quantum determinant would seem to place the latter outside the category of coboundary bialgebras.

5. Deformations of twisted quantum groups.

We are interested in the deformations of quantum groups; that is, the deformations of certain coboundary bialgebras inside the category of coboundary bialgebras. We pose

$$R(\epsilon) = R + \epsilon R_1 \quad (5.1)$$

and require that this matrix satisfy the Yang-Baxter relation to order ϵ . This is a linear condition on R_1 , but it is quadratic in R , which makes it complicated; besides, the cohomological interpretation is not clear to me. What follows is a search for an alternative formulation, invoking some ideas from $*$ -quantization already utilized by Drinfeld.

Suppose there is an operator F on $U \otimes U$ such that the (deformed) coproduct Δ of U can be expressed as

$$\Delta = F \Delta_0,$$

where Δ_0 is the standard coproduct (4.27). This is just the dual image of the formula (4.5) of a $*$ -product on A , where F is the bidifferential operator defined by $a*b = F(a, b)$. The simplest possibility is that F can be identified with an element of $U \otimes U$. Attempts to construct such an element in the case of the simplest quantum group were frustrated, and indeed it does not exist, as shown by K.M. Lau. His argument is this. For $u, v \in U$,

$$F \Delta_0(uv) = \Delta(uv) = \Delta(u)\Delta(v) = F \Delta_0(u) F \Delta_0(v)$$

If F is invertible we get $\Delta_0(u)\Delta_0(v) = \Delta_0(u)F \Delta_0(v)$ and thus $F = 1$. Therefore, more generally let F be an element of C_2^2 . According to ref. [13], there is no loss of generality in taking $F = \text{ad } N$, $N \in U \otimes U$, invertible,

$$\Delta(u) = \text{ad } N \Delta_0(u) = N \Delta_0(u) N^{-1}. \quad (5.2)$$

Calculations on the simplest quantum group confirms that such N exists. Note also that this formula is suggested by (4.4).

Consider the P-matrix for $U_{qq'}(gl_2)$,

$$P = \sum_i M_i^i \otimes M_i^i + (1 - q'/q) M_1^1 \otimes M_2^2 + q' M_1^2 \otimes M_2^1 + q^{-1} M_2^1 \otimes M_1^2. \quad (5.3)$$

(The P-matrix for twisted quantum $gl(n)$ is very similar and the calculation of N that follows generalizes immediately to that case.) with parameters q and q' . The associated starproduct is not defined, but the relations among the matrix elements of

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

implied by (4.30) are

$$\begin{aligned} a * b &= qb * a, \quad c * d = qd * c, \quad c * a = q'a * c, \quad d * b = q'b * d, \\ c * b &= qq'b * c, \quad a * d - d * a = (q - q')b * c. \end{aligned} \tag{5.4}$$

We ask for a pair of matrices N, \tilde{N} such that the product defined by

$$TT = \tilde{N}T * TN$$

is abelian. In other words, the $*$ -product would be obtainable from an abelian product by a twist

$$T * T = \tilde{N}^{-1}TTN^{-1},$$

this being equivalent to (5.2) if $\tilde{N} = N^{-1}$. The calculations can be done very efficiently with the aid of quantum planes and antiplanes, covariant and contravariant; the general solution is

$$N = 1 \oplus \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \oplus 1, \quad \tilde{N} = 1 \oplus \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{\delta} \end{pmatrix} \oplus 1,$$

with

$$\begin{aligned} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &= \begin{pmatrix} q + q' & q - q' \\ 0 & 2qq' \end{pmatrix} \begin{pmatrix} A & B \\ B & A \end{pmatrix} \\ \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{\delta} \end{pmatrix} &= \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{B} & \tilde{A} \end{pmatrix} \begin{pmatrix} 2qq' & q' - q \\ 0 & q + q' \end{pmatrix}. \end{aligned} \tag{5.5}$$

The coefficients A, B are arbitrary except $A^2 - B^2 \neq 0$. The basis used in $V \otimes V$ is $(11, 12, 21, 22)$, so for example $\beta = N_{12}^{21}$. The “opposite product” (dual to Δ') is characterized by similar relations, in which q, q' are replaced by $1/q, 1/q'$. The general expressions for the associated N-matrices are thus found by inverting the parameters, we denote them N', \tilde{N}' .

If the second (arbitrary) factors in N and in N' are taken equal, so that they cancel out in NN'^{-1} , then (with an appropriate renormalization) we find that the matrix that relates the two products, namely

$$NN'^{-1} = R = 1 \oplus (1/q) \begin{pmatrix} 1 & q - q' \\ 0 & qq' \end{pmatrix} \oplus 1, \tag{5.6}$$

agrees with (5.3) and satisfies the Yang-Baxter relation.

The formulas that relate the $*$ -product to the commutative product (that one can identify with the ordinary product of functions) depend on the choice of the arbitrary parameters A, B in the expression for the N-matrices. In the simplest case, when $A = \tilde{A} = 1/(q + q')$, $B = \tilde{B} = 0$, one has

$$\begin{aligned} a * b &= ab, \quad c * a = ca, \quad c * b = cb, \\ a * d &= ad + \frac{q - q'}{q + q'} cb. \end{aligned} \tag{5.7}$$

It should be noted that, unlike the relations (5.4), these last formulas do not by themselves define a product on A . They do not, for example, give any information about $a * (bc)$. There are two ways to complete the picture. In the first place one can revert to Eq(5.2); this equation determines a coproduct on U in terms of the standard coproduct Δ_0 , and by duality a product on A . The alternative is to choose a mapping that associates, to each ordinary polynomial in a, b, c, d , a $*$ -polynomial. Variations on this theme were discussed in Section 4D.

Formulas of the type (5.2) were introduced by Reshetikhin [5], and used, with an N-matrix constructed entirely from the Cartan subalgebra of \mathfrak{g} , to discover the generalized quantum groups that are generally known as twisted versions of the Drinfeld quantum groups. They can be used to calculate the further deformations of these twisted quantum groups. The result can be summarized as an isomorphism of the cohomology spaces of bialgebras and quantum groups. This completes the quantization of all the constant r-matrices for the simple, complex Lie algebras.

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